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A model of optimal development for an under-employed economy

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Abstract

The economy is described by a two-sector model, one representing 'Modern' activities and the other 'Traditional'. Two factors of production, labour and capital, each of whose productivities increases independently through time, are combined in fixed proportions to yield output: the output of the 'Traditional' sector is wholly consumed, whereas the output of the 'Modern' sector can be either consumed or invested, and if the latter, in either sector, at will.

Two control investments – the saving rate and the fraction of investment allocated to the 'Modern' sector – are available to secure the speediest elimination of unemployment.

Mathematically the dynamics of the model are reduced to a pair of differential equations representing the evolution of capital stocks per unit of augmented labour. The conditions that shadow prices of capital must meet through time are specified through the use of the Pontryagin-Hamilton Maximum Principle.

The solution is illustrated numerically for the case of the north-east of Thailand, suggesting that unemployment can be eliminated within a generation, if this is the sole objective of economic policy.

1. Introduction

A growth model is useful when it describes an economy with some accuracy, while indicating the likely outcomes of alternative policies. Two concerns of development economists in recent years are related to the above statement, the first to the accuracy of the description and the second to the direction of the policies. Our aim in writing this paper is to address these concerns, using as our vehicle an optimal growth model of a developing economy.

The descriptive phenomenon on which we wish to focus is *technical progress,* increasingly recognized as a major contributor to economic growth. Most well known are Denison's decomposition of factors contributing to the growth of the now-developed countries, but subsequent work by Denison himself and a collaborator (Denison and Chung 1976), and by Kim and Park (1985), have carried out decompositions for Japan and South Korea respectively. What is most relevant for the currently developing countries, and for those which are failing to develop, is their finding that the contribution of technical progress to the total growth of national income was much higher in these developing countries. If technical progress is measured by increases in output per unit of input, that is, as increases in *total factor productivity*, the average percentage rate of increase in Japan (1953–71) was nearly twice that of the average in ten now-developed countries, and in South Korea (1963–82) nearly one and a half times. If to the estimates of increases in output per unit of input is added the contribution of *education* of the labour force, the relative positions are unchanged, although the absolute numbers become more impressive—an average annual rate of increase of national income due to technical progress (in this broader sense) in Japan of 5.2 per cent, and in Korea of 3.6 per cent, versus an average rate of increase in the ten developed countries of 2.7 per cent (see Westphal 1986, for the comparisons).

Such high rates of growth of factor productivity are not wholly favourable, however, unless national income is growing at a still higher rate, enough higher to provide *employment* for a growing labour force. The experience in many developing countries (Morawetz 1974) is—simultaneously—increases in both industrial output and productivity and a rise in unemployment. The rise in unemployment leads us to focus on policies directed at reducing, or even eliminating, this scourge.

Several earlier models of optimal development admit to the existence of unemployment: those of Bose (1968, 1970), Dixit (1969), Kurz (1965), Stern (1972), and Stoleru (1965) have been available to development economists for many years. These models permit an evalution of the response of an economy to one or two policy instruments and an indication of the speed with which idle resources can be brought into play. Yet advances in efficiency with which resources can be utilized have a significant effect on the extent to which they can be employed (Enos and Park 1988), and none of the above models includes the steady improvement of technology which is a major feature in all economies.

The model which we shall formulate in this paper will attempt a description of an economy with underemployed resources and with technical progress. The policies which the model is designed to evaluate will be those by which the full employment of the country's resources may be most quickly attained. As in the earlier models, the optimal policy turns out to involve investing as much of current output as possible; unlike the earlier models, the interval to the attainment of full employment can be *quite short* and the sacrifice of current consumption need not be so great. For the economy desiring to maintain a certain minimal rate of consumption, technical progress relaxes the constraint on the nation's saving rate.

2. Assumptions underlying the model

Our growth model represents a *dual economy* in the sense of Lewis (1954). All economic activities fall neatly into one of two mutually exclusive categories: these categories or sectors will as usual be termed 'Modern' and 'Traditional'. The two sectors differ from each other in two major ways. The first is the intensity with which capital and labour resources are used to produce output, the Modern sector being relatively capital-intensive, the Traditional labour-intensive. Put simply, the capital–labour ratio in the former is, initially at least, higher than in the latter.

The second way in which the sectors differ is their ability to produce capital goods. We assume that investment arises in the Modern sector only. The outputs of the labour-intensive sector can only be consumed, whereas the outputs of the capital-intensive sector can serve as either expendable goods or investment goods; as expendable goods they are consumed, as investment goods saved.

Once a decision to save has been made, it is assumed that capital goods appropriate for either sector can be produced. Thus the Modern sector is assumed to be perfectly flexible. It is able to alter at will the composition of its output between consumption goods and capital goods, and between capital goods destined for installation in its own sector and capital goods destined for installation in the Traditional sector. Once allocated to a sector, however, capital goods will be assumed to remain there throughout their economic life. Rather than the *a priori* perfect substitutability of capital goods for labour given by the long-run *neoclassical* production function, we utilize a simpler fixed coefficient *Leontief* technology which in our opinion better models the reality of relatively short-term investment in actual capital goods in the presence of technical progress.

These assumptions require some explanation. We define investment broadly so that it includes not only the construction of new plant and equipment, but also the education of labour and the creation of institutions that facilitate the massing of resources, the standardization of production and the extension of markets. Undertaking investment thus involves 'modernizing' the economy.

Modernization is not carried out by all the economic agents of a country, nor everywhere within its boundaries. The impetus comes from a few well-educated and -connected individuals in business, finance, universities and institutes, and government. It is they who, by example and expression, promote changes in outlook. It is their commitment to economic development which can bring about institutional change. Their political power enables them to capture the surplus; their will and administrative control can assure its domestic use. They and their substantial assets—money, plant and equipment, 'know-how'—constitute the nucleus of the Modern sector of our model. The crucial importance of such individuals to development has been argued by Adelman and Morris (1968).

We have retained the usual nomenclature for sectors of a dual economy with some unease. Where authority rests with groups who, although they have long held power, are dedicated to developing the economy, the labels Modern and Traditional are not mutually exclusive: here the Traditional forces *are* the modernizing ones. Alternate names for the two sectors might be *Large-scale* and *Small-scale*, to indicate the extent of organization of economic activities; or *Collective* and *Individualistic*, to indicate not the ownership of property, but the degree of integration of talents and institutions. Even in purely capitalistic countries the private entrepreneur must have his own organization, not to mention far-reaching connections in government, finance and the media of communications, in order to ensure the success of his investments.

Implicit in this interpretation of the Modern sector of our model is the assumption that all foreign trade transactions are handled in the Modern sector. The production process in that sector thus subsumes trade of domestic product, agricultural or manufactured, for scarce foreign exchange and its allocation to the purchase of imported goods. We are forced for simplicity to *assume a trade balance and ignore foreign aid.* Even though exports may be produced in the Traditional sector, their earnings are transformed into capital goods through devices—marketing boards, exchange controls, licences, even expropriation—vested in the Modern sector. As a consequence of this approach, the production process of the Traditional sector is concerned only with agricultural and handcrafted goods for domestic use.

Implicit in this categorization of the Traditional sector is that all *additions* to capital made during the epoch the plan is in force originate outside the sector. Land, the major component of the capital stock of the Traditional sector, exists at the initial point in time, but the improvements to it during the running of the model—via such inputs as fertizilizers, access roads, agricultural extension services, even land reform—arise in the Modern sector. Since all observers argue that output in the Traditional sector grows rapidly or slowly depending upon the attention it receives from individuals in the Modern sector, it is from the resources available to the Modern sector that the Traditional sector's new capital must come.

A related point concerns our treatment of technical progress as *exogenous*. In our view there is little doubt that in advanced industrial economies technical progress is endogenous, in the sense that resources must be expended to secure productivity gains. Models of developed economies should properly take explicit account of the innovation process. However, nations of the Third World are in a different position. Their potential for productivity gain depends largely on an ability to raise capital to educate the work force and to manufacture under licence, or to purchase directly capital goods of an advanced design (see, e.g. Enos and Park 1988). Allowing for depreciation and replacement, the continual upgrading of technique at the micro level is fairly represented by exogenous technical progress at the macro level. Nonetheless, there is little reason to suppose that gains to the productivity of labour and capital will be *neutral*, or the same in the two sectors. Such a view is reflected in our model.

After formal statement in the next section, the model is transformed to a convenient mathematical form in §4 and its possible evolutions elaborated. In §5 the existence and structure of the optimal policy for full employment in minimal time are discussed, using the Pontryagin–Hamilton approach of optimal control theory. This policy is derived explicitly for estimates of the parameters based on data for the north-east of Thailand in §6. Section 7 includes a discussion of the compatibility of the minimal time policy with some other familiar planning goals and some suggestions for further development of the model.

3. Model statement

Of the variables, parameters and instruments that appear in our model, the *variables* (e.g. economic aggregates such as national income) can be expected to change with time, as a consequence of technological progress, government policies, etc. The *parameters* (e.g. a capital:output ratio) link the variables with one another, and are expected to stay constant over the planning horizon. The *instruments* are the controls available to society to achieve its goals and, like the variables, can vary with time.

The variables (capital Roman letters), parameters (lower case Greek letters) and instruments (lower case Roman letters) in the model are listed for convenience in Table 1.

Table 1. Model data

Variables	Modern sector	Traditional sector	Total economy
Output	Y_1	Y_2	Y
Consumption	C_1	C_2	C
Savings	S_1	–	–
Capital stock	K_1	K_2	–
Labour employed	L_1	L_2	–
Labour unemployed	–	–	–
Investment	I_1	I_2	I
Parameters			
Output:capital coefficient	α_1	α_2	–
Output:labour coefficient	β_1	β_2	–
Rate of growth of productivity of capital	γ_1	γ_2	–
Rate of growth of productivity of labour	λ_1	λ_2	–
Rate of depreciation of capital	δ_1	δ_2	–
Rate of growth of labour force	–	–	ν
Instruments			
Average savings rate	s	–	–
maximum attainable	$\overline{s}$	–	–
minimum permissible	$\underline{s}$	–	–
Allocation of investment	a	$(1-a)$	–
maximum proportion	$\overline{a}$	$(1-\overline{a})$	–
minimum proportion	$\underline{a}$	$(1-\underline{a})$	–

Our economy is comprised of two *sectors*, *Modern* (subscript 1) and *Traditional* (subscript 2), in each of which two *factors* of production, *capital* and *labour*, are employed in fixed proportions and with increasing productivity through time to produce a homogenous *good*:

$$Y_1 = \min\{\alpha_1 e^{\gamma_1 t} K_1,\ \beta_1 e^{\lambda_1 t} L_1\} \tag{3.1}$$

$$Y_2 = \min\{\alpha_2 e^{\gamma_2 t} K_2,\ \beta_2 e^{\lambda_2 t} L_2\}\,. \tag{3.2}$$

The variables in equations (3.1) and (3.2) and in subsequent equations are all functions of time, but where all variables in an equation refer to the same period, the time designation will be omitted. Implicit in our production functions (3.1) and (3.2), therefore, is the absence of a time lag between the application of the inputs and the attainment of the output.

Although all labourers will be assumed to be capable of working in *either* sector, the labour force need not be fully employed, L_3 being the *unemployed* residual.

Thus

$$L = L_1 + L_2 + L_3\,, \tag{3.3}$$

and the only constraint on L_1, L_2 and L_3 is that each be non-negative.

The labour force grows through time at a constant rate:

$$\dot{L} = \nu L\,. \tag{3.4}$$

Nothing in equations (3.1) through (3.4) distinguishes the Modern from the Traditional sector. The first distinction is made in the next equation,

$$S_1 = sY_1,\ \ 0 \le \underline{s} \le s \le \overline{s} \le 1\,. \tag{3.5}$$

The upper bound on the *savings rate* s expresses society's unwillingness to reduce consumption of the product of the Modern sector below a certain absolute amount; the lower bound, its determination to save at least a fraction of its output.

Only *part* of the output of the Modern sector is consumed, therefore, whereas *all* of the output of the Traditional sector is:

$$C_1 = (1-s)Y_1 \tag{3.6}$$

$$C_2 = Y_2\,. \tag{3.7}$$

We shall assume that new capital can be allocated to either sector, according to the desire of society:

$$I_1 = aI \tag{3.8}$$

$$I_2 = (1-a)I,\ 0 \le \underline{a} \le a \le \overline{a} \le 1\,, \tag{3.9}$$

and shall admit the possiblity of upper and lower bounds, $\overline{a}$ and $\underline{a}$, on the *allocation* instrument a.

A momentary *equilibrium* is imposed on savings and investment:

$$I = S_1 . \tag{3.10}$$

Once allocated, capital goods are *immobile*, and depreciate at constant rates:

$$\dot{K}_i + \delta_i K_i = I_i, \quad i = 1,2 . \tag{3.11}$$

Since we are considering an era when labour is less than fully employed, capital will be the scarce factor in the economy. Assuming that it will be used to full capacity, we can, by utilizing equations (3.1), (3.2), (3.5), (3.7), (3.8) and (3.9), determine the evolution of capital stock over time as

$$\begin{aligned} \dot{K}_1 &= (as\alpha_1 e^{\gamma_1 t} - \delta_1)K_1 \\ \dot{K}_2 &= (1-a)(s\alpha_1 e^{\gamma_1 t} K_1 - \delta_2)K_2 . \end{aligned} \tag{3.12}$$

The first equation in (3.12) is a simple description of the structure of a one-sector economy, à la Ramsay. Alternatively, by substituting (3.6), it may be written with consumption entering explicitly as

$$\dot{K}_1 = a\alpha_1 \, e^{\gamma_1 t} K_1 - \delta_1 K_1 - C_1 . \tag{3.13}$$

There only remains the formulation of the objective to which the economy is to be directed. Since our goal is the elimination of unemployment as quickly as possible, we put no value on production or consumption *per se*. Imagine a period of τ years elapsing before full employment is attained; then, in the τ^{th} year the conditions

$$L_1(\tau) + L_2(\tau) = L(\tau) \tag{3.14}$$

and

$$L_3(\tau) = 0 \tag{3.15}$$

are met, and in all previous years

$$L_1(t) + L_2(t) < L(t) . \tag{3.16}$$

Under these conditions the objective which the economy will have before it, the minimization of the interval to full employment, may be stated formally as

$$\text{maximize} \quad \int_0^{\tau} (-1)\, dt . \tag{3.17}$$

Thus, mathematically we have defined a free end point optimal control problem with terminal (linear) manifold specified by (3.14), (3.15) and state space constraints (typically nonbinding).

4. Development paths

It will be convenient to transform the basic equations (3.12) in such a way that they can be expressed in terms of capital per unit of effective labour, taking account of the labour-augmenting component of technological progress in the Modern sector. Specifying $k_i(t)$ as $K_i(t)e^{-\lambda_i t}/L(t)$, $i = 1,2$, will accomplish this transformation. The result will be of simpler form if we define two additional terms $\alpha^i(t) := \alpha_i e^{\gamma_i t}$ and $\eta_i := \nu + \delta_i$, $i = 1,2$, viz.

$$\dot{k}_1 = (\alpha^1 as - \eta_1)k_1$$
$$\dot{k}_2 = \alpha^1(1-a)sk_1 - \eta_2 k_2 . \tag{4.1}$$

These differential equations specify the rates of growth of capital stocks per unit of augmented labour in the *Modern* sector, that is, per unit of 'skilled hand' equivalents. Specifically, they describe the time rates of change of the transformed capital stock k_1 in the Modern sector and k_2 in the Traditional sector.

Using the fixed coefficient production functions (3.1) we may transform the *underemployment* constraint $L_3 \geq 0$ into the new variables as

$$\frac{\alpha^1}{\beta_1}k_1 + \frac{\alpha^2}{\beta_2}\, e^{(\lambda_1-\lambda_2)t}k_2 \leq 1 . \tag{4.2}$$

Our *programme* can now be posed as follows: Find time paths for the instruments which lie within our constraints and carry the system (4.1) in the shortest time τ from given non-negative initial conditions to final conditions $k_1(\tau)$, $k_2(\tau)$ which satisfy the equality version of (4.2); and describe the structure of the economy at the attainment of full employment. Note that equality in (4.2) constitutes a *moving* target unless technical change is purely labour-augmenting, that is, Harrod-neutral, and identical in the two sectors.

In the remainder of this section we shall examine the nature of feasible trajectories of capital formation in phase-space, that is, in the $k_1 - k_2$ plane. So long as capital is the scarce resource, equations (4.1) trace out all possible paths for the transformed variables k_1 and k_2. They are first-order differential equations with non-constant coefficients whose general solution (see, e.g. Struble 1962) for fixed initial values $k_1(0),\ k_2(0) \geq 0$ is

$$k_1(t) = e^{\int_0^t \alpha^1 as d\xi - \eta_1 t} k_1(0) \tag{4.3}$$

$$k_2(t) = e^{-\eta_2 t} \int_0^t \alpha^1(1-a)s\ e^{[\int_0^\xi \alpha^1 as d\zeta - (\eta_1 - \eta_2)\xi]} d\xi k_1(0) + e^{-\eta_2 t} k_2(0),$$

where e is the natural logarithm and the integrals are taken over time from 0 to t.

From the first equation of (4.3) it follows that capital stock in the Modern sector will grow or shrink exponentially according as $\alpha^1 as$ is greater or less than the constant rate η_1 at which the need for it increases. Inspection of the second equation shows a balance between exponential 'depreciation' at a rate η_2 (in response to physical exhaustion δ_2, labour's increasing productivity λ_1, and population growth ν) and the sum of initial capital stock and accumulated new capital produced in the Modern sector and delivered to the Traditional.

Equations (4.1) easily give a parametric differential equation for k_2, in terms of k_1 and t, as

$$\frac{dk_2}{dk_1} = \frac{\dot{k}_2}{\dot{k}_1} = \frac{a^1(1-a)sk_1 - \eta_2 k_2}{(a^1 as - \eta_1)k_1}, \tag{4.4}$$

which is valid for moments at which $\alpha^1 as$ is not equal to η_1. The solution of (4.2) is given by

$$k_2 = \frac{\alpha^1(1-a)s}{\alpha^1 as - \eta_1 + \eta_2} k_1 + c\, k_1^{-\eta_2/(\alpha^1 as - \eta_1)} \tag{4.5}$$

for moments at which $(a^1 as) - \eta_1$ is not equal to $-\eta_2$. For moments at which $(a^1 as) - \eta_1 = -\eta_2$ the solution is given by

$$k_2 = \frac{\alpha^1(1-a)s}{-\eta_2} k_1 \log k_1 + c\, k_1 . \tag{4.6}$$

In both cases c represents a constant determined by the initial conditions. When technical progress in the modern sector is *Harrod-neutral*, that is, when $\alpha^1 = \alpha_1$, equations (4.5) and (4.6) determine the trajectories of capital formation in the phase-plane for fixed values of the instruments. Rather than present a detailed analysis of this case (which is available from the authors on request) we shall concentrate on the general case in which there is augmentation in the Modern sector.

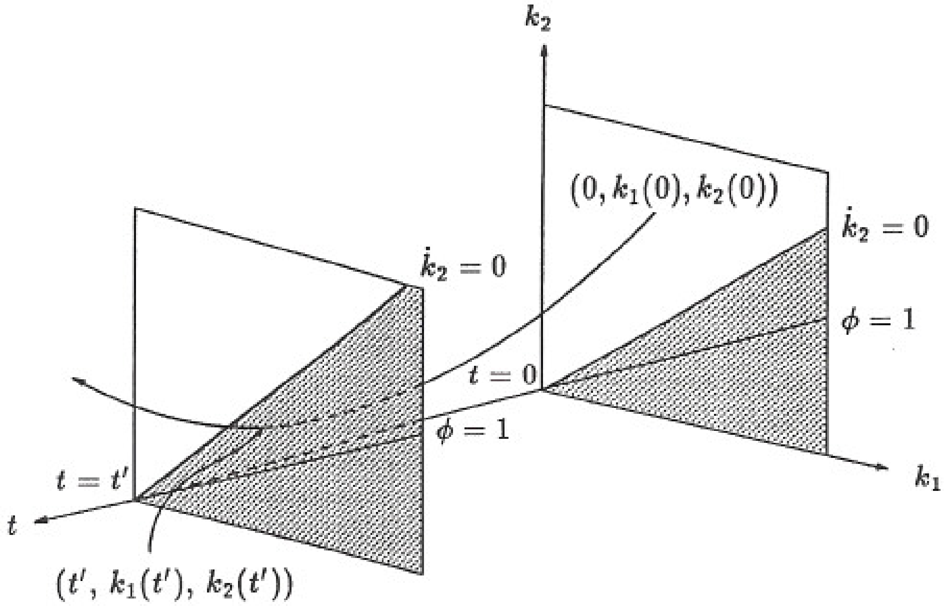

Figure 1

Path of capital formation in $t - k_1 - k_2$ space

In this case the system (4.1) is non-autonomous and the usual phase-plane analysis for fixed values of the instruments breaks down (Shell 1967). Nevertheless, in describing the nature of the curve described by equations (4.3) which passes through $(0, k_1(0), k_2(0))$ in $t - k_1 - k_2$ space, it will be useful to reinterpret some of the standard analysis in order to delimit the intersection of the curve with the sequence of $k_1 - k_2$ planes indexed by increasing t (see Fig. 1). (In the autonomous case the regions of each of these planes remain static and we may project the curve onto the plane at $t = 0$ to obtain a trajectory in a phase-plane diagram.) Recall that the two axes of the $k_1 - k_2$ planes measure the capital stocks per unit of augmented labour in the Modern and Traditional sectors respectively. The closer to the origin are the coordinates of a point identifying an economy, the *scantier* is its capital endowment. The closer to either axis, the *less balanced* is its capital composition.

The quadrant bounded by the positive k_1 and k_2 axes can be usefully subdivided into segments within which one or the other of the capital stocks grows or shrinks with t. The dividing lines are that at which $\dot{k}_1 = 0$ and that at which $\dot{k}_2 = 0$. Provided that $\alpha^1 as$ is greater than η_1, that is, that capital in the Modern sector is being 'added to' faster than it is being 'used up', $\dot{k}_1$ can be zero at t only if k_1 is zero. Given this proviso, the positive quadrant in the plane at t is the region of capital *accumulation* in the Modern sector. If $\alpha^1 as$

is less than η_1 at t, the entire quadrant will be a region of capital *decumulation* in the Modern sector.

The dividing line $\dot{k}_2 = 0$, unlike that of $\dot{k}_1 = 0$, bisects the positive $k_1 - k_2$ quadrant for each t. From the second equation of (4.1) we see that the slope of the line $\dot{k}_2 = 0$ is $\alpha^1(1-a)s/\eta_2$, and that $\dot{k}_2$ equals zero at moments at which the additions to the capital stock in the Traditional sector from the Modern sector are just equal to the depletions. Thus, given fixed values of a, s, η_1, and η_2, the line $\dot{k}_2 = 0$ passes through the origin. If α^1 were constant (i.e. if technological progress in the Modern sector were solely labour-augmenting) the slope of the line $\dot{k}_2 = 0$ would be fixed through time, but as α^1 is increasing (i.e. technological progress is both capital- and labour-augmenting) with increasing t, over time the line $\dot{k}_2 = 0$ rotates counter-clockwise with the origin as a pivot. With the productivity of the capital stock in the Modern sector steadily improving, a lower rate of injection is needed to maintain capacity in the Traditional.

A third dividing line in successive $k_1 - k_2$ planes is the locus of all points at which the proportional growth rates of the capital stocks in both sectors are equal. The slope of this line,

$$\phi := \frac{\dot{k}_2/k_2}{\dot{k}_1/k_2} = 1\,,$$

through the origin is easily obtained from equations (4.1) as

$$\frac{k_2}{k_1} = \frac{\alpha_1(1-a)s}{\alpha^1 as - \eta_1 + \eta_2}\,. \tag{4.7}$$

When $(\alpha^1 as) - \eta_1$ is greater than zero (the condition met in a developing as opposed to a stagnating country), the slope of the line $\phi = 1$ will be less than that of $\dot{k}_2 = 0$, as can be seen by comparing their equations. Like the line $\dot{k}_2 = 0$, the line $\phi = 1$ will rotate over time with capital-augmenting technological progress, although the direction of rotation is *not* unambiguous. If η_1 is *greater* than η_2, the line $\phi = 1$ will rotate *counter-clockwise* with time; if η_1 is *less* than η_2, *clockwise*. What we know of actual values for η_1 and η_2 would lead us to expect a *counter-clockwise rotation* in most cases. Indeed, η_i is defined as $\nu + \lambda_i + \delta_i$ and we should expect the rate of depreciation in the Modern sector to be higher than that in the Traditional, i.e. δ_1 to exceed δ_2. Since the two terms η_1 and η_2 are ultimately swamped by the third term in the denominator, $\alpha^1 as$ (i.e. $\alpha_1 e^{\gamma_1 t} as$), the angular velocity of rotation of the line $\phi = 1$ diminishes until in the limit its slope is the constant $(1-a)/a$.

The three contiguous regions whose boundaries, reading clockwise from the vertical to the horizontal, are the k_2 axis (with which the line $\dot{k}_1 = 0$ is coincident), the ray $\dot{k}_2 = 0$, the ray $\phi = 1$, and the k_1 axis, could be used for a *typology* of development. Lying between the k_2 axis and the ray $\dot{k}_2 = 0$ are those countries which at a given time have just begun to industrialize and are increasing their stock of capital in the Modern sector while decreasing that in the Traditional sector. These are countries in a truly '*underdeveloped*' state. Their rate of capital formation in the modern sector, as indicated by the control variable a (the portion of total investment allocated there), is high. Indeed, the Modern sector has prior claim on the new capital goods; i.e.

$$a = \frac{\eta_1}{\alpha^1 s} \text{ implies } \dot{k}_2 < 0\,,$$

or

$$k_2 > k_1(\alpha^1 s - \eta_1)/\eta_2\,.$$

Those countries which have passed over the boundary $\dot{k}_2 = 0$ into the region between $\dot{k}_2 = 0$ and $\phi = 1$ have progressed to the state where they are increasing their stocks of capital in both sectors simultaneously. They are '*developing*'. Any country so fortunate as to be in the third region, that lying between the line $\phi = 1$ and the k_1 axis, is '*developed*' in that it is increasing the stocks of capital in both sectors at a higher rate, and is likely to be facing a labour constraint.

The *breadth* of the two regions included between the lines $\dot{k}_2 = 0$ and the k_1 axis gives an indication of a country's *potential* for development. The larger the angle subtended by these two rays, the greater is the region within which capital is being accumulated simultaneously in both sectors. For an 'underdeveloped' country it is the breadth of the segment formed by the lines $\dot{k}_2 = 0$ and $k_1 = 0$ that indicates its potential at any given moment. Early in its history the wedge may be very thin, and if technological progress is slow, may remain so. If, however, the rate of technologicl progress in the Modern sector is rapid, the wedge will widen as time passes, for the upper boundary formed by $\dot{k}_2 = 0$ will be rotating steadily toward the k_2 axis.

Having examined the three regions of succesive k_1–k_2 planes, we show in Fig. 2 some trajectories of capital formation for fixed values of the instruments and initial positions. In these phase diagrams two cases are distinguished according as $(\alpha^1 as - \eta_1)$ is greater than or less than zero. In the *first* case, $(\alpha^1 as - \eta_1) > 0$, there is overall *growth*, for $\dot{k}_1$ is greater than zero throughout; in the *second*, $(\alpha^1 as - \eta_1) < 0$, *decline*. In the first case, Case I, a country following the upper trajectory would move in time from the region of underdevelopment

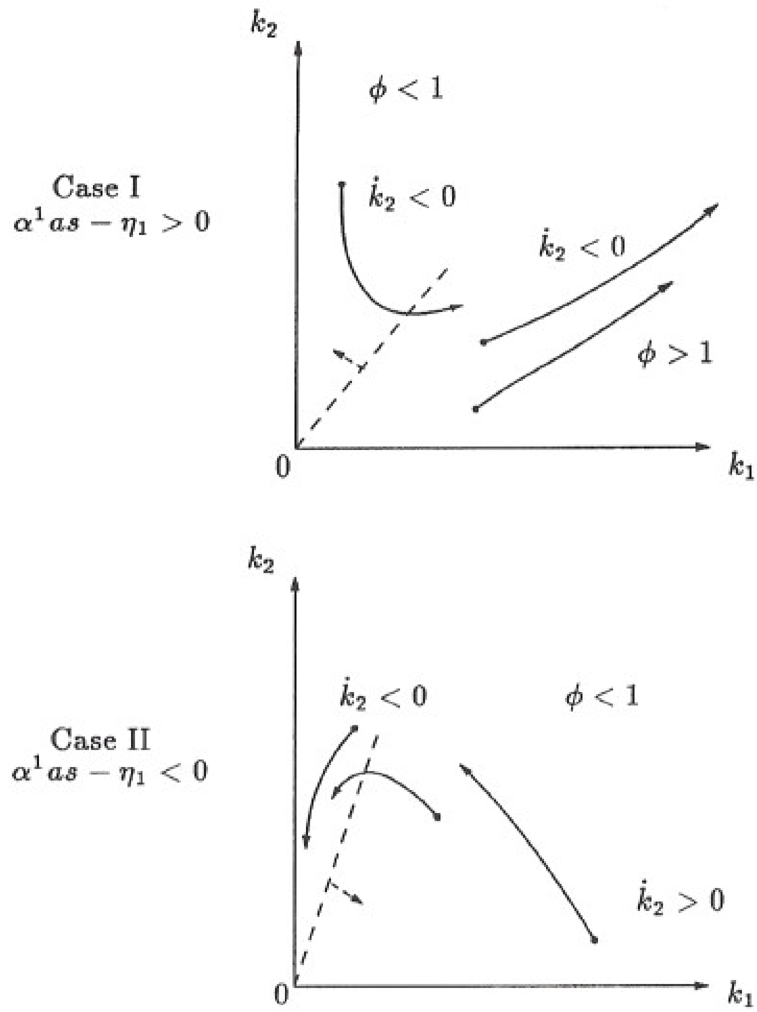

Figure 2

Trajectories of capital formation

($\dot{k}_1 > 0, \dot{k}_2 < 0$) to that of development ($\dot{k}_1 > 0, \dot{k}_2 > 0$). At the moment the trajectory hits the boundary ($\dot{k}_2 = 0$) between these two regions, its (transformed) capital stock in the Traditional sector would have struck its minimum. As illustrated by the second trajectory, once in the region of 'development' the trajectory would approach asymptotically the long-run balanced growth ray, $k_2 = [(1-a)/a]k_2$, given by the *asymptotic* value of $\phi = 1$. A country on the third trajectory is throughout 'endowing' its population with capital.

The phase diagram for Case II is appropriate where $(\alpha^1 as - \eta_1)$ is less than zero. It shows paths travelling to the left, for the rate at which capital is treated in the Modern sector is not high enough to maintain at its initial levels the stock of Modern capital per unit of augmented labour, nor, after a certain point, the stock of Traditional capital either. The region in which $\dot{k}_2 < 0$ is a degenerative one, but the region in which $\dot{k}_2 > 0$ is one of sufficient reallocation of new capital from the Modern to the Traditionl sector to permit the capital stock in the latter to increase. If it wished to build up its per capita stock of capital in the Traditional sector, an underdeveloped country might set its instruments so as to operate for a period in this lower region.

For $(\alpha^1 as - \eta_1) = 0$, vertical trajectories result, which rise or fall depending on the sign of $\dot{k}_2$.

The smoothness of the trajectories of capital stocks in the phase diagrams is deceptive, for no economy is likely to maintain fixed values of its policy instruments. Each trajectory in Fig. 2 is the consequence of choosing a *unique* and *fixed* set of values for the instruments. These values will of course be changed in order best to achieve the nation's objectives. Moreover, no account has been taken here of the labour constraint which will eventually bind and limit the accumulation of capital stocks per unit of augmented labour. Indeed, since labour augmentation is in the limit *infinite* in our model, all full-employment trajectories must eventually reach the *origin*.

5. Optimal policies

In our model the two policy instruments under the control of the government are s, the proportion of Modern sector output that can be saved (invested), and a, the fraction of total investment that is retained as capital in the Modern sector. Determining the optimal policy involves the choice of time paths for s and a. In order to facilitate the discussion in §7, consider the derivation of an optimal policy with respect to the general integral *objective*

$$\text{maximize} \int_0^{\tau} w(k_1, k_2, a, s, t)\, dt \tag{5.1}$$

over the (finite or infinite) *horizon* T, where τ denotes the *time* to reach a terminal manifold (which may be the entire state space). (We assume that the integral exists for all feasible policies, for the moment, and that w is continuously partially differentiable in all its arguments.)

Since modern control theory is adequately covered in textbooks (see, e.g. Luenberger 1969; Hadley and Kemp 1971; Fleming and Rishel 1975; Kamien and Schwartz 1981) and has been described for a non-mathematical audience in simple terms (Dorfman 1969), we proceed directly with its application to our system, whose *state variables* are the transformed capital stocks, k_1 and k_2, and whose *controls* are s and a.

We define two *dual* variables, p_1 and p_2, which may be interpreted as current ***shadow prices*** of the capital stocks in the Modern and Traditional sectors respectively relative to the *objective function* w_1 and set up the *Hamiltonian* in terms of the function w and the shadow prices as

$$H := w + p_1 \dot{k}_1 + p_2 \dot{k}_2 \,. \tag{5.2}$$

In terms of the capital stocks it is given by

$$H = w + p_1[\alpha^1 as - \eta_1]k_1 + p_2[\alpha^1(1-a)sk_1 - \eta_2 k_2]\,, \tag{5.3}$$

utilizing equations (4.1). The Hamiltonian function expresses the sum of the current objective function value w and the increment to it due to the current rates of change of the capital stocks, as measured by the shadow prices.

Taking partial derivatives of (5.3) yields the equation system

$$\begin{aligned} \dot{k}^0 &= \frac{\partial H^0}{\partial p} = A(t)k^0 \\ \dot{p}^0 &= \frac{\partial H^0}{\partial k} = -\frac{\partial w^0}{\partial k} - A'(t)p^0\,, \end{aligned} \tag{5.4}$$

which describes the *optimal* evolution of state variables and shadow prices in vector form. Here p^0 is the vector of optimal shadow prices, k^0 the vector of optimal stocks and A the matrix of coefficients of the equation system (4.1). The first equation of (5.4) is simply the differential equations (4.1), describing the evolution of the capital stocks, in vector form. The first term on the right-hand side of the second equation of (5.4) is the *gradient* $-\partial w^0/\partial k$ of the objective integrand with respect to the vector of capital stocks. The second term $A'(t)p^0$ involves the transpose of the coefficient matrix of (4.1). *Optimal* controls a^0, s^0 must *maximize* the Hamiltonian function at each moment of time, i.e.

$$H^0 = H^0(a^0, s^0, t) \geq H(a, s, t) = H\,, \text{ for all } t\,. \tag{5.5}$$

Provided these momentary maxima exist for each t, the *Pontryagin–Hamilton optimality conditions* (5.5) may be stated in *differential* form in terms of the partial derivatives of the Hamiltonian with respect to the controls,

$$\frac{\partial H}{\partial a} = \frac{\partial w}{\partial a} + [p_1 - p_2]\alpha^1 s k_1$$

$$\frac{\partial H}{\partial s} = \frac{\partial w}{\partial s} + [a p_1 + (1-a)p_2]\alpha^1 k_1 \,. \tag{5.6}$$

Namely: $\partial H^0/\partial a$ is less than, equal to, or greater than zero at moments at which $a^0 = \underline{a}$, $\underline{a} < a_0 < \overline{a}$ or $a^0 = \overline{a}$, respectively, while $\partial H^0/\partial s$ has identical properties at moments at which $s^0 = \underline{s}$, $\underline{s} < s^0 < \overline{a}$ or $s_0 = \overline{s}$, respectively. If w is concave in a and s for each t from 0 to τ, these conditions, together with a *transversality condition* for finite $\tau \leq T$, are both *necessary and sufficient* to characterize an optimal policy for given initial capital stock, even though the corresponding Hamiltonian is not concave in the usual sense. (The transversality conditions place a restriction on terminal shadow prices.) Sufficiency is most easily demonstrated by integrating (5.5) from 0 to τ and applying the usual saddle-point argument of nonlinear programming to the result (see, e.g. Dempster 1974). We note that the maximum value H^0 of the Hamiltonian will *change* with time according to

$$\frac{\partial H^0}{\partial t} = \frac{\partial w^0}{\partial t} + [a p_1^0 + (1-a)p_2^0]\gamma_1 \alpha^1 s k_1 \,, \tag{5.7}$$

unless technical progress in the Modern sector is Harrod-neutral and w is independent of time.

The objective of minimizing the time τ to full employment may be treated as a special case of (5.1) in which $w := -1$ and $\tau \leq T := \infty$. Then the second equation of (5.4) describing the optimal evolution of shadow prices yields the differential equations,

$$\dot{p}_1 = -(\alpha^1 a s - \eta_1)p_1 - \alpha^1(1-a)s\, p_2$$
$$\dot{p}_2 = \eta_2 p_2 \,, \tag{5.8}$$

the *adjoint* system to (4.1). For given initial shadow prices of capital, $p_1(0)$ and $p_2(0)$, the solution of (5.7) is given by

$$p_1(t) = e^{-\int_0^t \alpha^1 a s d\xi + \eta_1 t}[p_1(0) - \int_0^T \alpha^1(1-a)s e^{\int_0^\xi \alpha^1 a s d\zeta - (\eta_1 - \eta_2)\xi} d\xi \, p_2(0)]$$

$$p_2(t) = e^{\eta_2 t} p_2(0) \,. \tag{5.9}$$

We may deduce the nature of the *time-optimal policy* from (5.9) and the Pontryagin–Hamilton conditions using the appropriate version of (5.5),

$$\frac{\partial H}{\partial a} = [p_1 - p_2]\alpha^1 s k_1$$
$$\frac{\partial H}{\partial s} = [a p_1 + (1-a) p_2]\alpha^1 k_1 \,, \tag{5.10}$$

and a suitable *transversality* condition. This condition is given by

$$\frac{\alpha^1(\tau)}{\beta_1}\, p_1^0(\tau) - \frac{\alpha^2(\tau)}{\beta_2}\, e^{(\lambda_1-\lambda_2)\tau} p_2^0(\tau) = 0 \,, \tag{5.11}$$

which expresses the fact that at the attainment of full employment at τ, when the capital stocks satisfy the equation constraint (4.2), the optimal capital stock and shadow price trajectories must be orthogonal. *Existence* of an optimum follows from the continuity of the time to full employment in the controls (substitute (4.3) in the equation version of (4.2) and use the implicit function theorem) over the compact constraint set (Fleming and Rishel 1975).

It follows from the Pontryagin–Hamilton conditions and the expression for $\partial H/\partial a$ in (5.10) that the Hamiltonian will be maximized if $a^0 = \overline{a}$ at moments when $p_1^0 > p_2^0$; and $a^0 = \underline{a}$ at moments when $p_1^0 < p_2^0$. (The trajectories determined by (5.9) imply that $p_1 = p_2$ only *momentarily* if s^0 is constant over time.) For initial conditions of interest, the expression for $\partial H/\partial s$ in (5.10) implies similarly that $s^0 = \overline{s}$ as long as $a^0 p_1^0 + (1-a^0) p_2^0 > 0$. For these initial conditions $p_1^0(0) > 0$, as new capital must surely be accumulated in the Modern sector to reach full employment quickly. Whether or not a switch in a^0 from $\overline{a}$ to $\underline{a}$ occurs depends on the given initial conditions, but it is clear from the first equation of (5.9) that, if a switch is to occur, $p_2^0(0)$ must be positive. The second equation of (5.9) implies that p_2^0 remains positive, so that it follows from the transversality condition (5.11), that p_1^0 must remain positive and hence $s^0 = \overline{s}$ throughout.

Thus the optimal policy to attain full employment in minimal time from a typical initial position, in which capital in the Modern sector is scarce, has the following form: *save at the maximal rate throughout the period and allocate initially to the Modern sector as much new capital as possible. At a subsequent time, depending on initial capital stocks and parameter values, as much new capital as possible should be allocated to the Traditional sector.*

Such an optimal policy is calculated numerically for a typical example in the next section. In §7 this 'bang-bang' policy is related to policies optimal with respect to more general objectives.

Table 2. Initial conditions and parameter values for North-east Thailand (1960). Source: Enos (1970)

	Initial conditions	Modern sector	Traditional sector	Total economy
$K_i(0)$	(monetary units)	2.94×10^9	32.20×10^9	35.14×10^9
$L_i(0)$	(persons)	0.59×10^6	8.05×10^6	8.64×10^6
$L_3(0)$	(persons)	–	–	0.34×10^6
$k_i(0)$	(thousand monetary units per capita)	0.326	3.57	–
	Parameter values	**Modern sector**	**Traditional sector**	**Total economy**
ν	(fraction per year)	–	–	0.02
γ_i	"	0.03	0.02	–
λ_i	"	0.05	0.01	–
δ_i	"	0.05	0.00	–
α_i	"	0.4	0.25	–
β_i	"	2.0	1.0	–
$\overline{s}$	(fraction)	0.5	–	–
$\underline{s}$	"	0.3	–	–
$\overline{a}$	"	0.9	–	–
$\underline{a}$	"	0.3	–	–

6. A numerical example

Environmental conditions are expressed as initial points and parameter values in a growth model. For empirical purposes, statistical estimates must be made of the numerical values of the variables and parameters linking the variables at the beginning of the planning period and the latter estimates are *assumed* to remain valid over the planning period.

Our example is based on the north-east of Thailand, for which the environment has been adequately described for the purposes of a detailed computer simulation. (In a sense, our mathematical model represents the simplified essence of the much more complicated computer model.) It is a region embracing one-third of the land area of Thailand and containing an equal portion of its population. In terms of the model, its description is given for the initial year in Table 2 (Enos 1970).

The sectors are divided in such a way that, with the exception of government, education and other social overheads, all the economic activities taking place in the countryside and within the villages fall within the Traditional sector; the remainder fall within the Modern. The Modern sector is relatively small, containing initially 8 per cent of the total capital stock and employing 6 per cent of the labour force. The initial capital:labour ratio (β_1/α_1) in the Modern sector is 5.0, while that in the Traditional sector (β_2/α_2) is 4.0. Actual output per man in the initial period was approximately 820 Baht per worker in the Traditional sector and double this figure in the Modern. The annual rates of increase of productivity of both capital and labour are higher in the Modern sector (3 per cent and 5 per cent respectively) than in the Traditional (2 per cent and 1 per cent respectively). We take these as instantaneous rates which correspond to slightly higher annual rates.

In 1960 unemployment in the real economy was thought to be fairly low, 4.2 per cent of the labour force. It was feared that it might rise in the future, since the rate of growth of the population had increased with the eradication of malaria, whereas the rate of creation of new jobs in the Traditional sector had fallen with the exhaustion of virgin land.

Growth in the economy repesented by the model can take place by two means, technological progress and investment. Through technological progress a given level of resource employment in production becomes steadily more productive. Through investment the scarce resource, capital, is increased. Technological progress increases output directly, while investment increases output indirectly through increased employment of labour. It was assumed that the economy can save between 30 per cent and 50 per cent of its annual output, a not impossibly high percentage in a poor country (Riskin 1986). We further assume that up to 90 per cent of total investment but no less than 30 per cent can take place in the Modern sector. The ranges of the instruments are therefore

$$\begin{aligned} \underline{s} &= 0.30 \leq 0.50 = \overline{s} \\ \underline{a} &= 0.30 \leq 0.90 = \overline{a}\,. \end{aligned} \tag{6.1}$$

In order to determine the optimal policy to carry the model from the initial conditions of Table 1 to full employment in minimal time, we must examine behaviour over time of the *full-employment* line

$$\frac{\alpha^1}{\beta_1}\,k_1 + \frac{\alpha^2}{\beta_2}\,e^{(\lambda_1-\lambda_2)t}k_2 = 1\,. \tag{6.2}$$

Since technical progress is more rapid in the Modern sector, i.e. $\lambda_1 > \lambda_2$, the line of equation (6.2) moves towards the origin with decreasing

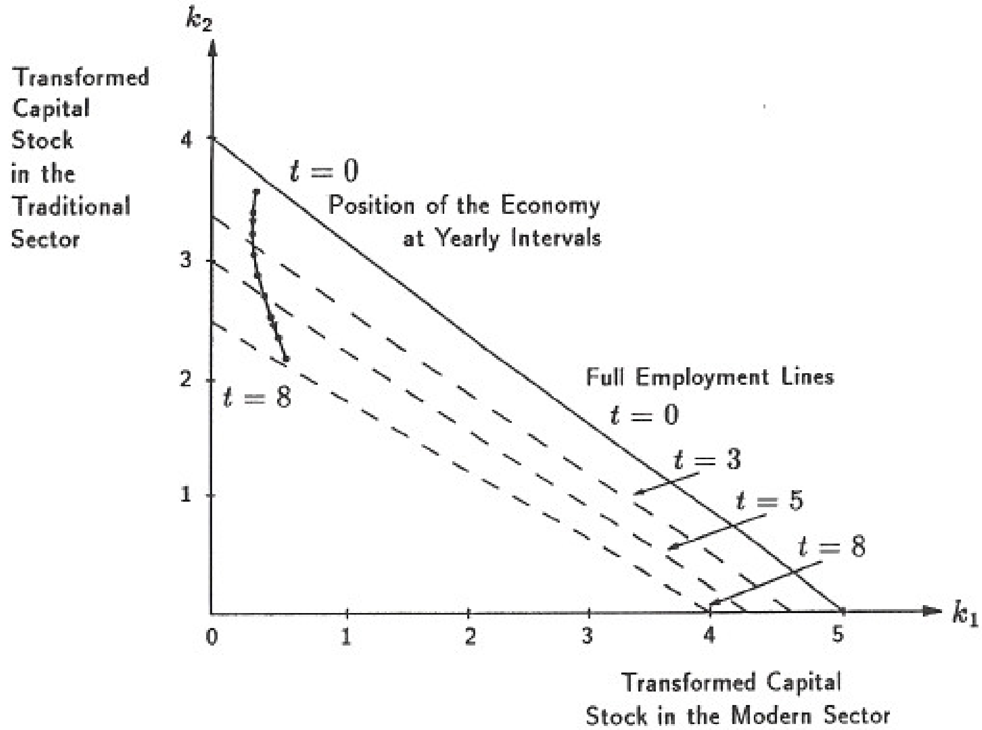

Figure 3
Trajectory of transformed capital stocks under a programme of maximal investment in the modern sector

slope. This is illustrated in Fig. 3: from its position at $t = 0$ (the solid line) it has reached by years $t = 3$, 5 and 8 the positions indicated by the three successive dashed lines.

We have seen in the previous section that the determination of the optimal minimal-time policy for typical 'underdeveloped' initial capital stocks reduces to the determination of the optimal switch point of maximal investment from the Modern sector to the Traditional, i.e. $s = \overline{s}$, $a = \overline{a}$ to $s = \overline{s}$, $a = \underline{a}$. Since for our parameter values progress is possible, i.e. $\alpha^1 \overline{a}\, \overline{s} - \eta_1 > 0$ for all t, the model economy will follow the trajectory illustrated at annual intervals in Fig. 3 if maximal investment is assigned to the Modern sector throughout (cf. Fig. 2, Case I). The dots for the first 7 years lie to the left of the full employment line; following this trajectory to full employment the economy would have excess labour for 7.8 years. (Of course, once full employment has been reached, a trajectory along the full employment line, in which both factors of production are in short supply, must be followed.) By switching to the trajectory along which maximal investment is assigned to the Traditional sector after 5.0 years, full employment is reached in 7.0 years, as illustrated in Fig. 4.

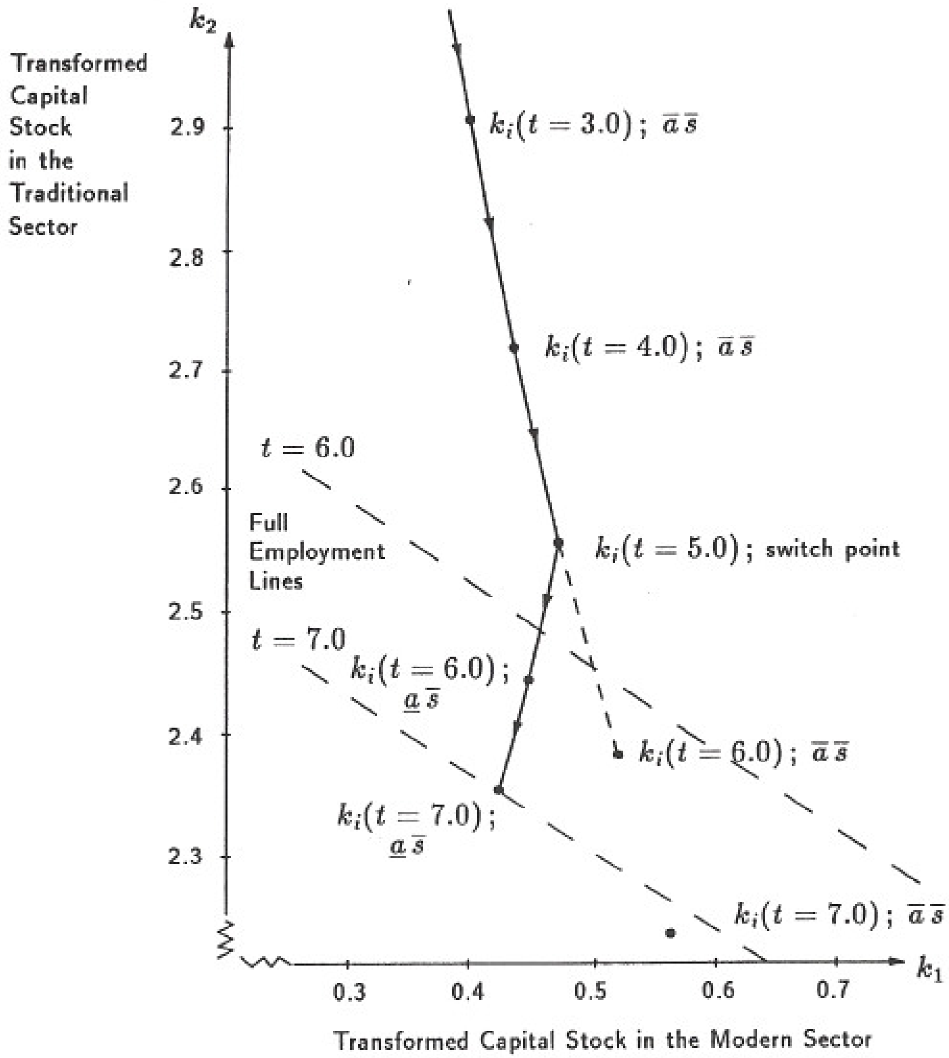

Figure 4
Trajectories of transformed capital stocks in the terminal years of the optimal programme

With $s = \bar{s}$ and $a = \bar{a}$, for our parameter values $\alpha^1 \underline{a}\bar{s} - \eta_1 < 0$, the transformed capital stock falls in both the Traditional and Modern sectors (cf. the left-most arrow in Fig. 2, Case II). These trajectories were calculated using (4.3) for k_1 and the standard difference equation approximation to the differential equation (3.1) for k_2 (with a step length of 0.2 years) and guessing at the switch point. (Truncation of the infinite series in a solution for k_2 similar to (6.4) below was not attempted.)

That the switch point after 5.0 years is optimal at this level of accuracy may be verified as follows. It suffices to demonstrate the existence of shadow prices p_1^0 and p_2^0 such that at the optimal switch point τ'

$$p_1(\tau') = p_2(\tau'), \tag{6.3}$$

and at the minimal time to full employment τ^0 the transversality condition (5.10) is satisfied. Now for fixed values s and a of the instruments the solution (5.9) may be integrated from t_1 to t_2 for initial values $p_1(t_1)$ and $p_2(t_1)$ as

$$\begin{aligned} p_1(t_2) &= e^{[e^{\frac{-as\alpha_1}{\gamma_1}}(e^{\gamma_1 t_1} - e^{\gamma_1 t_2}) + \eta_1(t_1 - t_2)]} \\ &\qquad \times \{p_1(t_1) - \alpha_1(1-a)se^{-as\alpha_1/\gamma_1} f(t_1, t_2) p_2(t_1)\} \quad (6.4) \\ p_2(t_2) &= e^{\eta_2(t_2 - t_1)} p_2(t_1), \end{aligned}$$

where

$$f(t_1, t_2) := \frac{[e^{(\eta_2-\eta_1)t_2} - e^{(\eta_2-\eta_1)t_1}]}{\eta_2 - \eta_1} + \frac{as}{\gamma_1} \sum_{i=1}^{\infty} \frac{e^{(\gamma_1 i + \eta_2 - \eta_1)t_2 - (\gamma_1 i + \eta_2 - \eta_1)t_1}}{i!\,(\gamma_1 i + \eta_2 - \eta_1)}.$$

Using (6.4) with $t_1 = 0$, $t_2 = \tau'$, $s = \bar{s}$ and $a = \bar{a}$ in (6.3) determines the optimal initial shadow price ratio $p_1^0(0)/p_2^0(0)$ in terms of τ'. Hence taking initial shadow prices in the ratio determined by $\tau' = 5.0$ the resulting shadow prices (calculated from the difference equation version of (5.8) with the same step length as above) will be equal at τ' and must satisfy the transversality condition (5.10) at 7.0, if 5.0 is the optimal switch point. This was found to be the case at the given level of accuracy when the infinite series in (6.4) was truncated after 25 terms to yield the optimal initial shadow price ratio 1400 for Modern sector over Traditional sector investment.

(After integration of the capital stocks (4.3) similar to (6.4) above, the full employment line (6.2) yields one equation in τ' and τ^0 in terms of the given initial capital stocks. Setting $t_1 = \tau'$, $t_2 = \tau^0$, $s = \bar{s}$ and $a = \underline{a}$ in (6.4) and using the previously found expressions for $p_1^0(\tau')$ and $p_2^0(\tau')$ yields another expression for the optimal initial shadow price ratio in terms of τ' and τ^0. Equating the two gives a second equation in τ' and τ^0 which can in principle be used in conjunction with the first to determine τ' and τ_0. In practice, these equations are highly nonlinear and the easiest approach is that described above.)

Economically, the optimal programme consists of building up first the Modern sector, where capital is created, and then the Traditional sector, where each unit of new capital creates more employment of labour. The switch in the allocation of new capital from the Modern to the Traditional sector occurs when the capital stock in the former has grown to the extent that it can generate sufficient additions to the capital stock of the latter. Switching either earlier or later would postpone the attainment of full employment. In the first event, the Modern sector would not be large enough to provide the necessary investment in the Traditional; in the second, the period of capital transfer to the Traditional sector would be too short to take advantage of its greater labour employment. Objectives under which allocation of new capital is gradually, rather than abruptly, switched are discussed in the next section. For the moment, note that our model economy is never rich enough for capital in both sectors to grow simultaneously. According to the typology suggested earlier, it has been 'underdeveloped' throughout the seven-year planning period. In fact, although fully employed, it is likely to remain underdeveloped for some long time thereafter. We shall return to this point in §7.

In order to point up the effect of increasing productivity of capital in our model, it is interesting to note that in the case of purely *labour-augmenting* technical progress, at the rates of Table 1, full employment is optimally reached only after 33.3 years, with switch point after 25.5 years. Not all productivity increase is a boon to all segments of society. With *no productivity increases*, full employment is optimally reached in 11.0 years (switching after 7.4 years), although at much lower levels of per capita income and consumption. This latter time to full employment is much shorter than the former because, of course, the labour-augmenting effects of capital formation are not present.

7. Conclusion

In this paper we have constructed a mathematical model of a dual economy in which the productivities of inputs (labour and capital in both sectors) increase steadily with time. We postulated that 'modernizing' investment occurs in only one of the two sectors, but that once the investment is undertaken, the resulting capital goods can be constructed for either sector. Two policy instruments were identified: the fraction of the Modern sector's output which is set aside for investment, and the fraction of investment which is allocated to either sector. By applying these instruments the policy makers direct the economy towards its goal.

Since most of the economies of the Third World are characterized by unemployed resources, we chose as our environment one in which labour is not fully occupied and as our objective the quickest possible elimination of unemployment. Mathematically this involved reducing the model to a pair of differential equations representing the evolution of capital stocks per unit of augmented labour, defining shadow prices for the two capital stocks and specifying the conditions that the shadow prices must meet through time so as to fulfil the objective of minimum time to full employment. The technique used was the Pontryagin–Hamilton Maximum Principle.

The solution of the model was illustrated for the case of the north-east of Thailand, a region with a moderate rate of growth of the labour force and a meagre endowment of capital in the Modern sector. Given these initial conditions, the optimal programme consisted of maintaining the highest possible savings rate throughout, and allocating as much as possible of the investment goods created by savings to the Modern sector, for a period of 5.0 years, and thereafter to the Traditional sector, for the final 2.0 years. Even though the savings rate, one-half of the output of the Modern sector, is kept very high and unemployment is eliminated as quickly as possible, the unemployment rate first rises from 4.2 per cent to a maximum of 4.6 per cent, before it begins to fall. Three years after the beginning of the optimal programme it is the same as initially, and when the switch occurs two years later it is still 3.5 per cent.

The chief difference between our model and others allowing unemployment is the incorporation of technological progress. This has several effects. First, the time necessary to eliminate unemployment, in an economy for which this goal is paramount, is reduced. In the case we investigated, technological progress with typical rates of factor augmentation enabled full employment to be reached in a little less than two-thirds the time. The increasing productivity of the scarce resource, capital, enabled increased investment, and hence employment, even though the increasing productivity of labour required increasing capital to maintain the level of employment.

A second effect of technological progress is that total consumption may no longer act as a constraint on the allocation of current output to investment. In models without technological progress (e.g. Stoleru 1965), the optimal programme (of maximum savings) may have to be relaxed if per capita consumption falls below its initial value. In our example consumption per capita increased throughout, although at a very low rate in the first few years. However, this feature is unlikely to hold with lower rates of technological progress, higher possible rates of savings, or a proportionately larger Modern sector.

A third effect of adding general exogenous technological progress to a dual economy model is to *negate* the importance of *balanced* economic growth. It is not that the optimal path to full employment under the minimal time objective is one of unbalanced growth, one sector being favoured at the expense of the other, but that even with more general objectives, or during any subsequent regime of full employment, growth must move along an *unbalanced* path. The only exceptions to this statement concern the case of Harrod-neutral technical progress with identical rates of increase of labour productivity in both sectors, and the objective of maintaining balanced growth itself.

With different objective functions the optimal programme will of course vary: no *general* prescription for optimal development exists. Indeed, for the example of §6 it might be argued that although full employment was reached nearly a year sooner with the time-optimal policy, it would be better to maintain the policy of maximal investment in the Modern sector ($s = \bar{s}$, $a = \bar{a}$) throughout the period of unemployment, so as to avoid having to reallocate new capital from one sector to the other. With respect to more general objectives involving explicit distributional assumptions, this policy might leave the economy better poised for subsequent full-employment growth. For certain values of the parameters of our model, if full employment is reached after building capital stock in the Modern sector to minimal levels, it will be impossible subsequently to maintain full employment, except perhaps for a relatively short period during which capital stock is accumulated in the Traditional sector. Another period of investment in the Modern sector will then be needed. It is perhaps the existence of this 'underdeveloped cycle' that prompts planners to adopt goals that would be expressed in our model as maximizing a weighted average of terminal capital stocks over a fixed planning period, normally shorter than the minimal time to full employment. The resulting objective function w is a function of both instruments, but leads to an optimal policy of the same form as the minimal time policy with modified shadow prices.

In our opinion, more general planning considerations are better incorporated in our model by the maximization of an infinite integral of a suitably discounted planners' utility function u. This should take explicit account of per capita consumption from the two sectors separately, and of unemployment. Even neglecting the usual difficulties with discounting and existence, the problems of deducing the *explicit* structure of the optimal policy (using the methods of §5) and of handling the transition from under- to full employment are formidable. (Bose (1970) has solved these problems for a similar but simpler model with a linear utility function and no technical progress.) Due to the appearance of all the variables shown in (5.1) in u (in general, nonlinearly), the optimal policy will no longer involve abrupt switches of the instruments and there will be a period when the optimal trajectory of transformed capital stocks will be governed by (5.4) and the equations resulting from equating the expressions in (5.6) to zero. Quantitatively, full employment will be reached later than in minimal time, but at a time and with a potential for maintaining a balance between Traditional and Modern sectors which will depend on the relative sizes of the partial derivatives of u.

It should be pointed out to readers who object to fixed coefficients of production that a trivial extension of our model to allow *temporally* dependent α_i, β_i in (3.1–2) will not alter our *qualitative* conclusions, but will, on the other hand, make these coefficients completely compatible with (representative firm) optimizing behaviour of the two producing sectors against neoclassical production functions (cf. Becker 1981). Moreover, under apparently reasonable conditions on the planners' utility function and the time behaviour of (optimal) production coefficients in this suggested generalization of our model, the (optimized) underdeveloped 'cycle' referred to above may even become *chaotic*. (For a demonstration of such behaviour in a *discrete time* two-sector neoclassical representative producer-consumer model of similar mathematical structure to ours, see Boldrin 1987.)

Another interesting extension to our model would be to deal with a multi-sector version with different rates of saving in the sectors (cf. Dasgupta 1969). This would alleviate the very careful identification of all activities that might comprise the Modern sector which would be needed to use our model in its present form for practical planning purposes. The assumption of fixed coefficients might also gain more general acceptance in a multi-sector version.

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